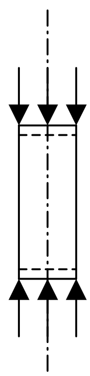
6 INTRODUCTION TO COLUMN BUCKLING

1.0 INTRODUCTION AND BASIC CONCEPTS

There are many types of compression members, the column being the best known. Top chords of trusses, bracing members and compression flanges of built up beams and rolled beams are all examples of compression elements. Columns are usually thought of as straight vertical members whose lengths are considerably greater than their cross-sectional dimensions. An initially straight strut or column, compressed by gradually increasing equal and opposite axial forces at the ends is considered first. Columns and struts are termed "*long*" or "*short*" depending on their proneness to buckling. If the strut is "short", the applied forces will cause a compressive strain, which results in the shortening of the strut in the direction of the applied forces. Under incremental loading, this shortening continues until the column "squashes". However, if the strut is "long", similar axial shortening is observed only at the initial stages of incremental loading. Thereafter, as the applied forces are increased in magnitude, the strut becomes "*unstable*" and develops a deformation in a direction normal to the loading axis. (See Fig.1). The strut is in a "*buckled*" state.

Buckling behaviour is thus characterized by deformations developed in a direction (or plane) normal to that of the loading that produces it. When the applied loading is increased, the buckling deformation also increases. Buckling occurs mainly in members subjected to compressive forces. If the member has high bending stiffness, its buckling resistance is high. Also, when the member length is increased, the buckling resistance is decreased. Thus the buckling resistance is high when the member is "*stocky*" (i.e. the member has a high bending stiffness and is short) conversely, the buckling resistance is low when the member is "*slender*".

Structural steel has high yield strength and ultimate strength compared with other construction materials. Hence compression members made of steel tend to be slender. Buckling is of particular interest while employing steel members, which tend to be slender, compared with reinforced concrete or prestressed concrete compression members. Members fabricated from steel plating or sheeting and subjected to compressive stresses also experience local buckling of the plate elements. This chapter introduces buckling in the context of axially compressed struts and identifies the factors governing the buckling behaviour. The local buckling of thin flanges/webs is not considered at this stage. These concepts are developed further in a subsequent chapter.



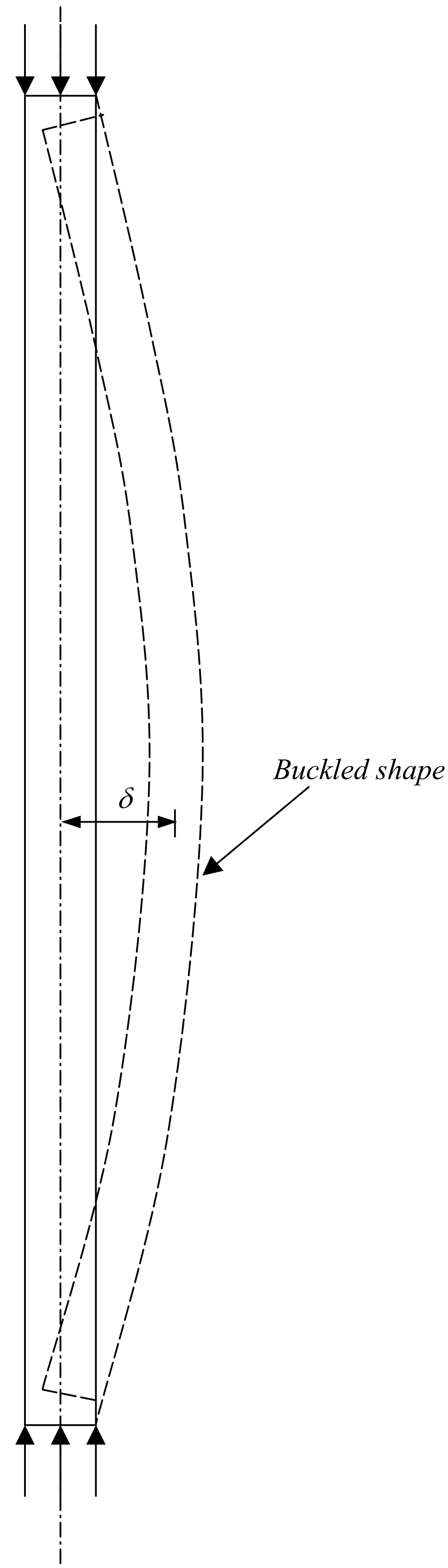


Fig 1: "short" vs "long" columns

2.0 ELASTIC BUCKLING OF AN IDEAL COLUMN OR STRUT WITH PINNED END

To begin with, we will consider the elastic behaviour of an idealized, pin-ended, uniform strut. The classical Euler analysis of this problem makes the following assumptions.

- the material of which the strut is made is homogeneous and linearly elastic (i.e. it obeys Hooke's Law).
- the strut is perfectly straight and there are no imperfections.
- the loading is applied at the centroid of the cross section at the ends.

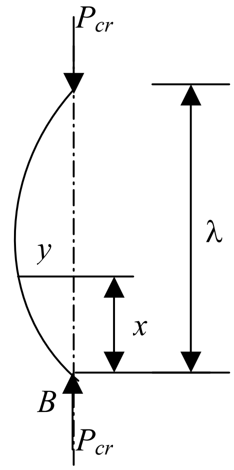


Fig. 2 Column Buckling

We will assume that the member is able to bend about one of the principal axes. (See Fig. 2). Initially, the strut will remain straight for all values of P, but at a particular value $P = P_{cr}$, it buckles. Let the buckling deformation at a section distant x from the end B be y.

The bending moment at this section = $P_{cr}.y$

The differential equation governing the small buckling deformation is given by

$$-EI\frac{d^2y}{dx^2} = P_{cr}.y$$

The general solution for this differential equation is

$$y = A_1 \cos x\sqrt{\frac{P_{cr}}{EI}} + B_1 \sin x\sqrt{\frac{P_{cr}}{EI}}$$

where A_1 and A_2 are constants.

Since $y = 0$ when $x = 0, A_1 = 0.$

when $x = \lambda$, $y = 0$;

Hence $B_1 \sin \lambda \sqrt{\dfrac{P_{cr}}{EI}} = 0$

Either $B_1 = 0$ or $\sin \lambda\sqrt{\dfrac{P_{cr}}{EI}} = 0$

$B_1 = 0$ means $y = 0$ for all values of x (i.e. the column remains straight).

Alternatively $\sin\lambda\sqrt{\dfrac{P_{cr}}{EI}} = 0$

This equation is satisfied only when

$$\lambda\sqrt{\frac{P_{cr}}{EI}} = 0, \pi, 2\pi, ...$$

$$P_{cr} = \frac{\pi^2 EI}{\lambda^2}, \frac{4\pi^2 EI}{\lambda^2} ..... \frac{n^2\pi^2 EI}{\lambda^2}$$

where n is any integer.

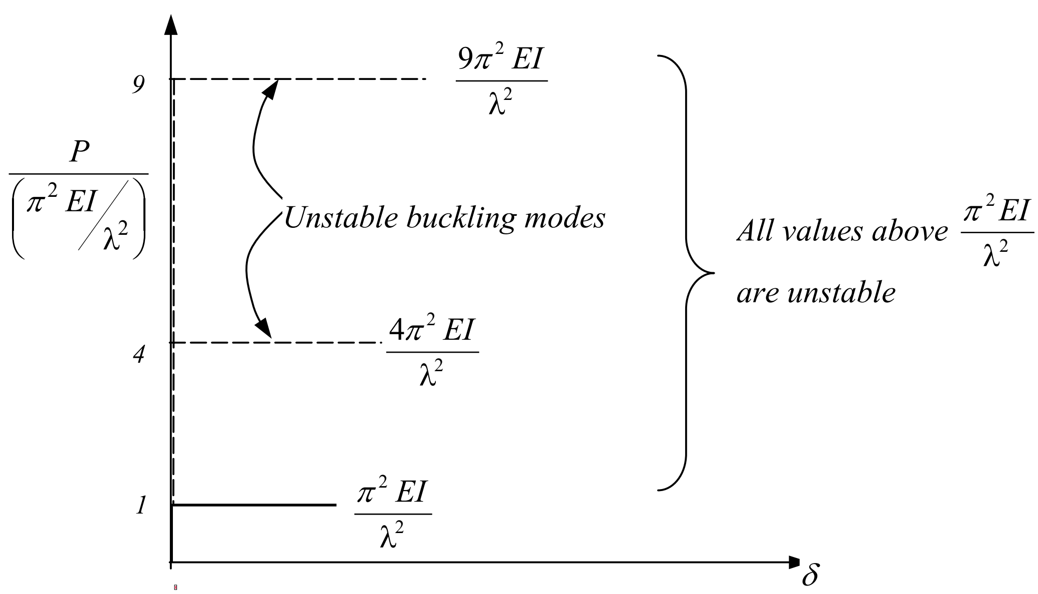


Fig. 3 Buckling load Vs Lateral deflection Relationship

While there are several buckling modes corresponding to $n = 1, 2, 3, \ldots$, the lowest *stable* buckling mode corresponds to $n = 1$. (See Fig. 3).

The lowest value of the critical load (i.e. the load causing buckling) is given by

$$P_{cr} = \frac{\pi^2 EI}{\lambda^2} \tag{1}$$

Thus the Euler buckling analysis for a " straight" strut, will lead to the following conclusions:

1. The strut can remain straight for all values of P.
2. Under incremental loading, when P reaches a value of $P_{cr} = \frac{\pi^2 EI}{\lambda^2}$

   the strut can buckle in the shape of a half-sine wave; the amplitude of this buckling deflection is indeterminate.
3. At higher values of the loads given by $\frac{n^2 \pi^2 EI}{\lambda^2}$ other sinusoidal buckled shapes (n half waves) are possible. However, it is possible to show that the column will be in unstable equilibrium for all values of $P > \frac{\pi^2 EI}{\lambda^2}$

   whether it be straight or buckled. ***This means that the slightest disturbance will cause the column to deflect away from its original position. Elastic Instability may be defined in general terms as a condition in which the structure has no tendency to return to its initial position when slightly disturbed, even when the material is assumed to have an infinitely large yield stress.*** Thus

   $$P_{cr} = \frac{\pi^2 EI}{\lambda^2} \tag{2}$$

   represents the maximum load that the strut can usefully support.

It is often convenient to study the onset of elastic buckling in terms of the mean applied compressive stress (rather than the force). The mean compressive stress at buckling, σ_{cr} , is given by

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{A\lambda^2}$$

where A = area of cross section of the strut.

If r = radius of gyration of the cross section, then $I = Ar^2$,

Hence, $$\sigma_{cr} = \frac{\pi^2 E r^2}{\lambda^2} = \frac{\pi^2 E}{(\lambda / r)^2} = \frac{\pi^2 E}{\lambda^2} \tag{3}$$

where λ = the slenderness ratio of the column defined by $\lambda = \lambda / r$
The equation $\sigma_{cr} = (\pi^2 E)/\lambda^2$, implies that the critical stress of a column is inversely proportional to the square of the slenderness ratio of the column (see Fig. 4).

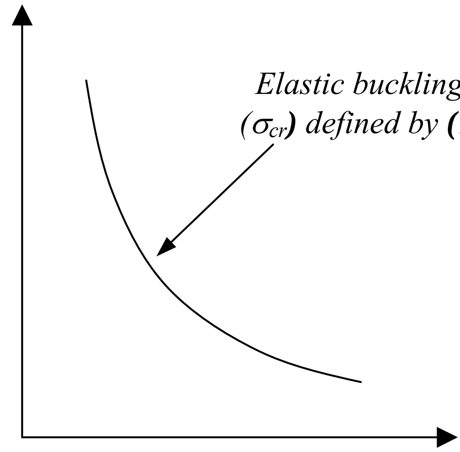


Fig. 4 Euler buckling relation between σ_{cr} and λ

3.0 STRENGTH CURVE FOR AN IDEAL STRUT

We will assume that the stress-strain relationship of the material of the column is defined by Fig. 5. A strut under compression can therefore resist only a maximum force given by $f_y.A$, when plastic squashing failure would occur by the plastic yielding of the entire cross section; this means that the stress at failure of a column can never exceed f_y , shown by A-A^1 in Fig. 6(a).

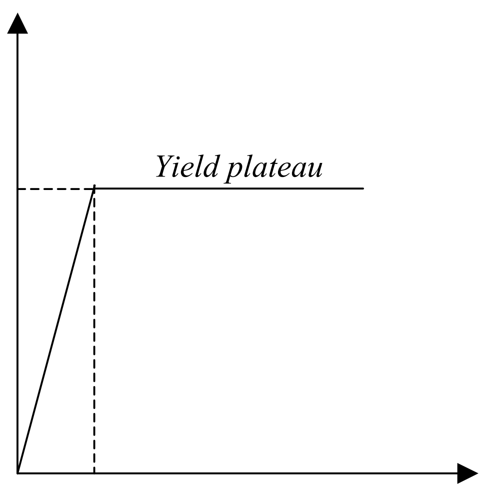


Fig. 5 Idealized elastic-plastic relationship for steel

From Fig. 4, it is obvious that the column would fail by buckling at a stress given by $\left(\dfrac{\pi^2 E}{\lambda^2}\right)$

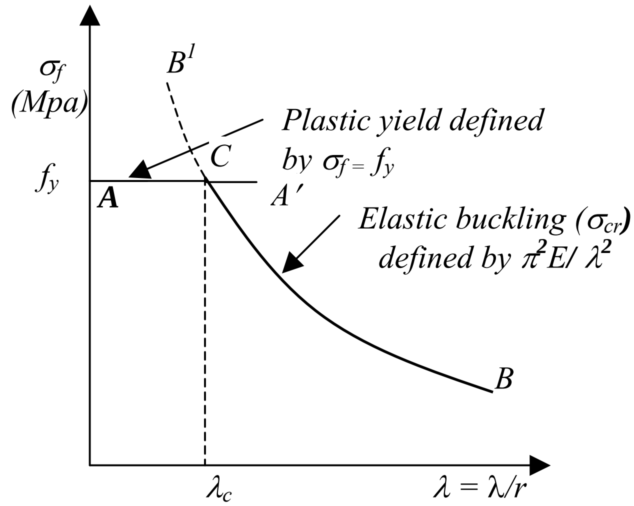


Fig. 6(a) Strength curve for an axially loaded initially straight pin-ended column

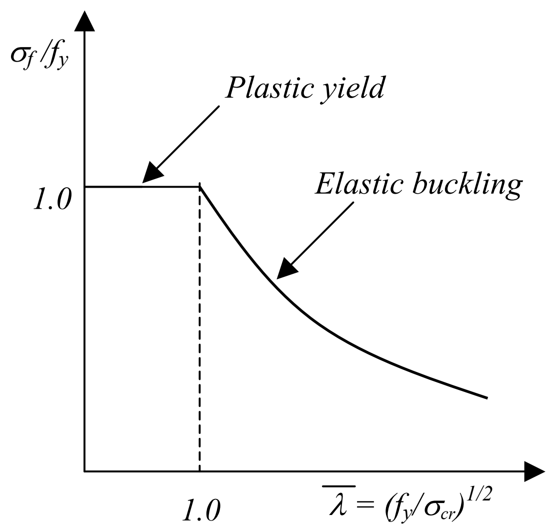


Fig. 6(b) Strength curve in a non-dimensional form

This is indicated by B-B^1 in Fig. 6(a), which combines the two types of behaviour just described. The two curves intersect at C. Obviously the column will fail when the axial compressive stress equals or exceeds the values defined by ACB. In the region AC, where the slenderness values are low, the column fails by yielding. In the region CB, the failure will be triggered by buckling. The changeover from yielding to buckling failure occurs at the point C, defined by a slenderness ratio given by λ_c and is evaluated from

$$f_y = \frac{\pi^2 E}{\lambda_c^2}$$

$$\lambda_c = \pi \sqrt{\frac{E}{f_y}} \qquad (5)$$

Plots of the type Fig. 6(a) are sometimes presented in a non-dimensional form illustrated in Fig. 6(b). Here (σ_f / f_y) is plotted against a generalized slenderness given by

$$\bar{\lambda} = \lambda / \lambda_c = \sqrt{f_y / \sigma_{cr}} \tag{6}$$

This single plot can be employed to define the strength of all axially loaded, initially straight columns irrespective of their E and f_y values. The change over from plastic yield to elastic critical buckling failure occurs when $\bar{\lambda} = 1$ (i.e. when $f_y = \sigma_{cr}$), the corresponding slenderness ratio $\left(\frac{\lambda}{r}\right)$ *is* $\pi \sqrt{\frac{E}{f_y}}$

4.0 STRENGTH OF COMPRESSION MEMBERS IN PRACTICE

The highly idealized straight form assumed for the struts considered so far cannot be achieved in practice. Members are never perfectly straight; they can never be loaded exactly at the centroid of the cross section. Deviations from the ideal elastic plastic behaviour defined by Fig. 5 are encountered due to strain hardening at high strains and the absence of clearly defined yield point. Moreover, residual stresses locked-in during the process of rolling also provide an added complexity.

Thus the three components, which contribute to a reduction in the actual strength of columns (compared with the predictions from the "ideal" column curve) are

(i) initial imperfection or initial bow.
(ii) Eccentricity of application of loads.
(iii) Residual stresses locked into the cross section.

4.1 The Effect of Initial Out-of-Straightness

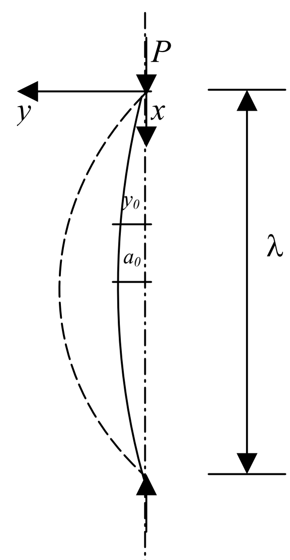


Fig. 7 Pin-ended strut with initial imperfection

Fig. 7 shows a pin-ended strut having an initial imperfection and acted upon by a gradually increasing axial load. As soon as the load is applied, the member experiences a bending moment at every cross section, which in turn causes a bending deformation. For simplicity of calculations, it is usual to assume the initial shape of the column defined by

$$y_0 = a_0 \sin\frac{\pi x}{\lambda} \tag{7}$$

where a_o is the maximum imperfection at the centre, where $x = \lambda / 2$. Other initial shapes are, of course, possible, but the half sine-wave assumed above corresponding to the lowest node shape, represents the greatest influence on the actual behaviour, hence is adequate.

Provided the material remains elastic, it is possible to show that the applied force, P, enhances the initial deflection at every point along the length of the column by a multiplier factor, given

$$\frac{1}{1-(P/P_{cr})} \tag{8}$$

The deflection will tend to infinity, as P is increased to P_{cr} as shown by curve-A, see Fig. 8(a).

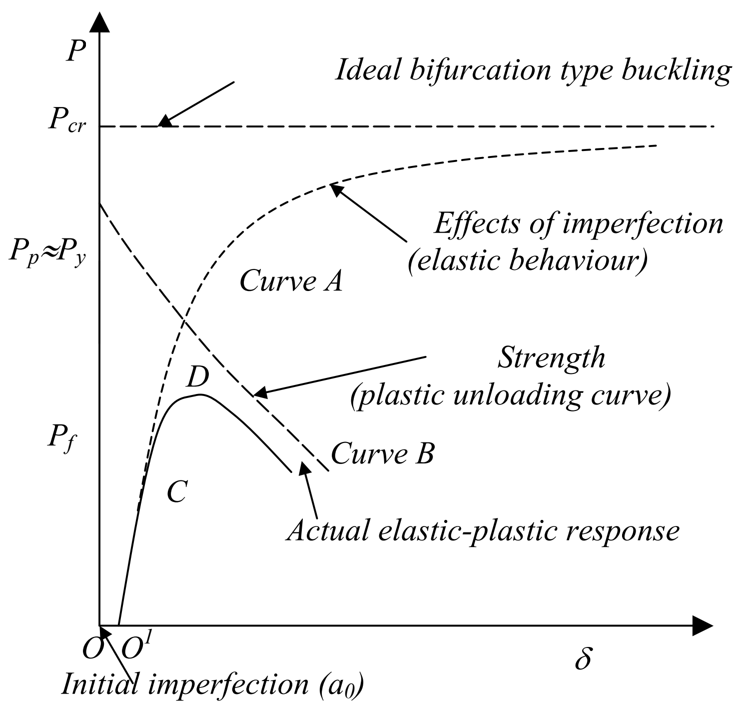


Fig. 8(a) Theoretical and actual load deflection response of a strut with initial imperfection

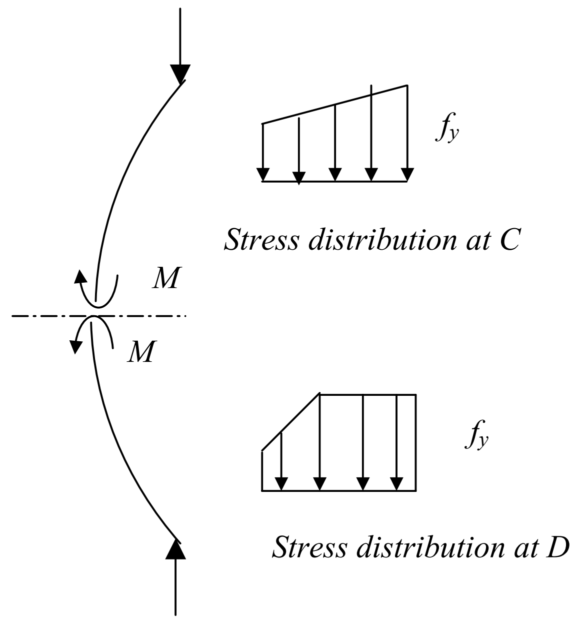


Fig. 8(b) Stress distributions at C and D

As the deflection increases, the bending moment on the cross section of the column increases. The resulting bending stress, ($M\ y/I$), on the concave face of the column is compressive and adds to the axial compressive force of P/A. As P is increased, the stress on the concave face reaches yield (f_y). The load causing first yield [point C in Fig. 8 (*a*)] is designated as P_y. The stress distribution across the column is shown in Fig. 8(*b*). The applied load *(P)* can be further increased thereby causing the zone of yielding to spread across the cross section, with the resulting deterioration in the bending stiffness of the column. Eventually the maximum load P_f is reached when the column collapses and the corresponding stress distribution is seen in Fig. 8 (*b*). The extent of the post-first-yield load increase and the section plastification depends upon the slenderness ratio of the column.

Fig. 8(*a*) also shows the theoretical rigid plastic response curve *B*, drawn assuming $P_{cr} > P_p$ (Note $P_p = A.\, f_y$). Quite obviously P_{cr} and P_p are upper bounds to the loads P_y and P_f. If the initial imperfection a_o is small, P_y can be expected to be close to P_f and P_p. If the column is stocky, P_{cr} will be very large, but P_p can be expected to be close P_y. If the column is slender, P_{cr} will be low and will often be lower than P_p or P_y. In very slender columns, collapse will be triggered by elastic buckling. Thus, for stocky columns, the upper bound is P_p and for slender columns, P_{cr} .If a large number of columns are tested to failure, and the data points representing the values of the mean stress at failure plotted against the slenderness (λ) values, the resulting lower bound curve would be similar to the curve shown in Fig. 9.

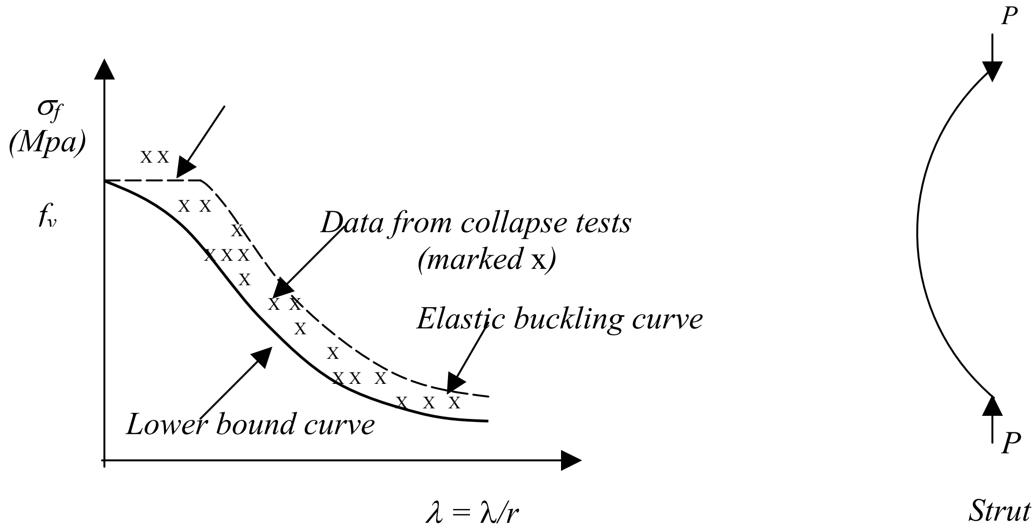


Fig 9: Strength curves for strut with initial imperfection

For very stocky members, the initial out of straightness – which is more of a function of length than of cross sectional dimensions – has a very negligible effect and the failure is by plastic squash load. For a very slender member, the lower bound curve is close to the elastic critical stress (σ_{cr}) curve. At intermediate values of slenderness the effect of initial out of straightness is very marked and the lower bound curve is significantly below the f_y line and σ_{cr} line.

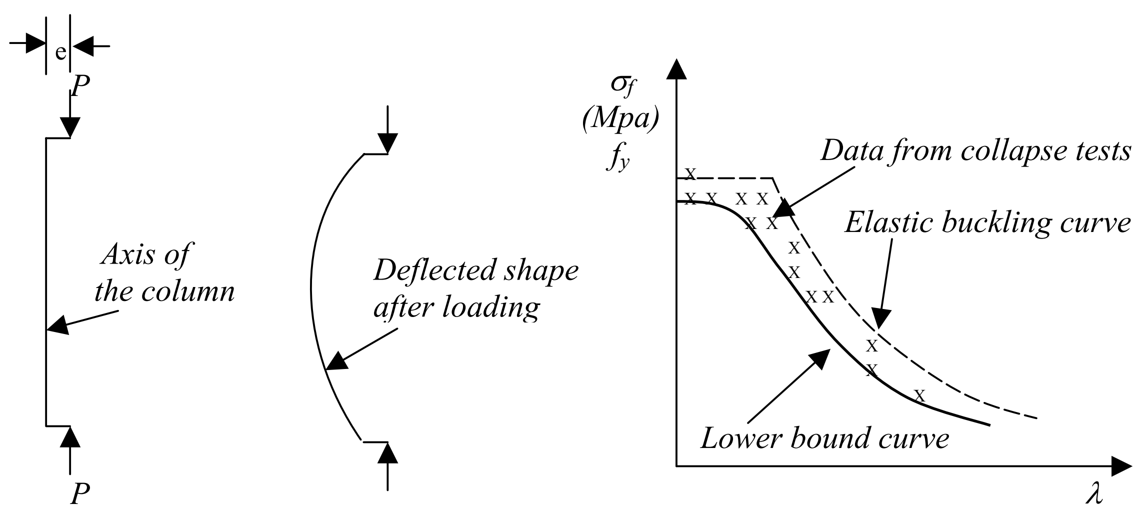


Fig. 10 Strength curve for eccentrically loaded columns

4.2 The Effect of Eccentricity of Applied Loading

As has already been pointed out, it is impossible to ensure that the load is applied at the exact centroid of the column. Fig. 10 shows a straight column with a small eccentricity (e) in the applied loading. The applied load (P) induces a bending moment ($P.e$) at every cross section. This would cause the column to deflect laterally, in a manner similar to the initially deformed member discussed previously. Once again the greatest compressive stress will occur at the concave face of the column at a section midway along its length. The load-deflection response for purely elastic and elastic-plastic behaviour is similar to those described in Fig. 8(*a*) except that the deflection is zero at zero load.

The form of the lower bound strength curve obtained by allowing for eccentricity is shown in Fig. 10. The only difference between this curve and that given in Fig. 9 is that the load carrying capacity is reduced (for stocky members) even for low values of λ.

4.2 The Effect of Residual Stress

As a consequence of the differential heating and cooling in the rolling and forming processes, there will always be inherent residual stresses. A simple explanation for this phenomenon follows. Consider a billet during the rolling process when it is shaped into an I section. As the hot billet shown in Fig. 11(*a*) is passed successively through a series of rollers, the shapes shown in 11(*b*), (*c*) and (*d*) are gradually obtained. The outstands (*b*-*b*) cool off earlier, before the thicker inner elements (*a*-*a*) cool down.

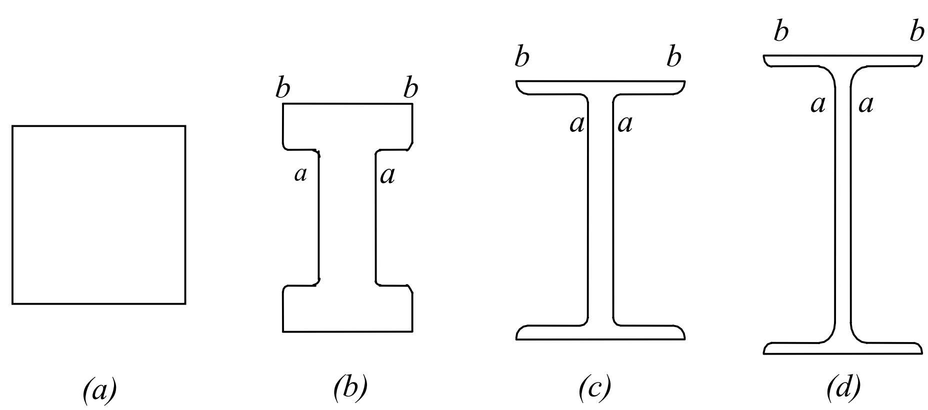
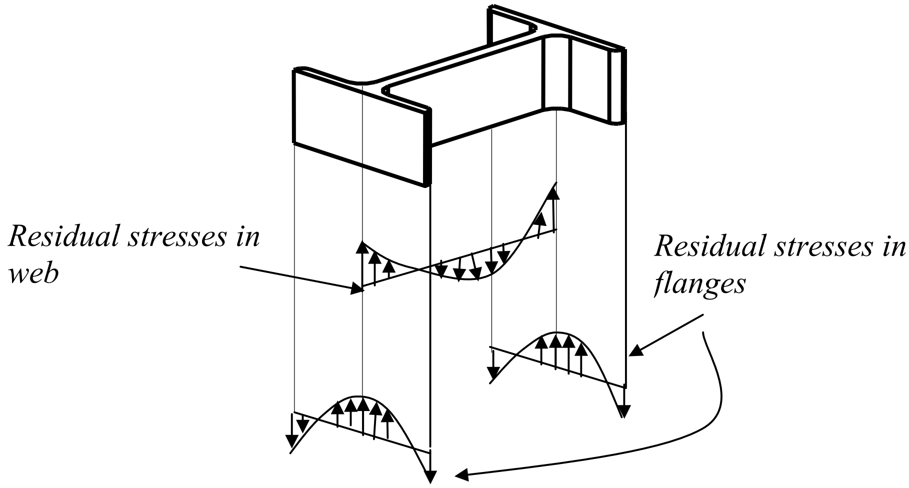


Fig. 11 Various stages of rolling a steel girder

As one part of the cross section (*b*-*b*) cools off, it tends to shrink first but continues to remain an integral part of the rest of the cross section. Eventually the thicker element (*a*) also cool off and shrink. As these elements remain composite with the edge elements, the differential shrinkage induces compression at the outer edges (*b*). But as the cross section is in equilibrium – these stresses have to be balanced by tensile stresses at inner location (*a*). The tensile stress can sometimes be very high and reach upto yield stress. The compressive stress induced due to this phenomenon is called "*residual compressive stress*" and the corresponding tensile stress is termed " *residual tensile stress*".



Residual stresses distribution (no applied load)

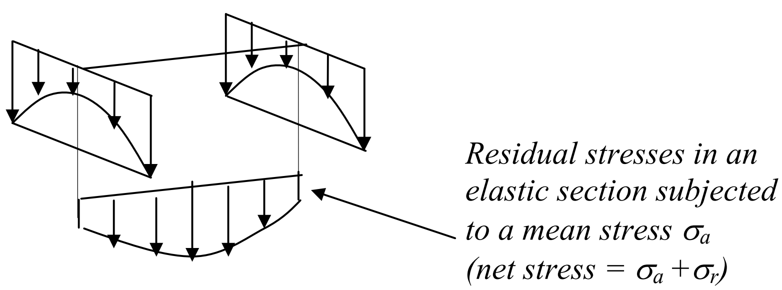


Fig. 12 The influence of residual stresses

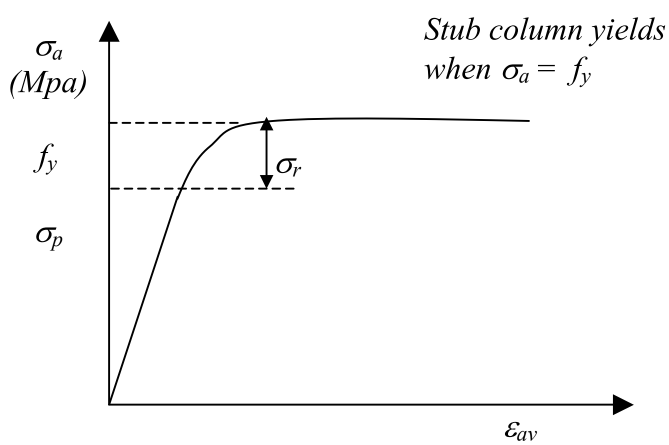


Fig. 13 Mean axial stress vs mean axial strain in a stub column test

Consider a short compression member (called a "*stub column*", Fig. 12(*a*) having a residual stress distribution as shown in Fig. 12 (*b*). When this cross section is subjected to an applied uniform compressive stress (σ_a) the stress distribution across the cross section becomes non-uniform due to the presence of the residual stresses discussed above. The largest compressive stress will be at the edges and is $(\sigma_a + \sigma_r)$

Provided the total stress nowhere reaches yield, the section continues to deform elastically. Under incremental loading, the flange tips will yield first when $[(\sigma_a + \sigma_r) = f_y]$. Under further loading, yielding will spread inwards and eventually the web will also yield. When $\sigma_a = f_y$, the entire section will have yielded. The relationship between the mean axial stress and mean axial strain obtained from the stub column test is seen in Fig. 13.

Only in a very stocky column (i.e. one with a very low slenderness) the residual stress causes premature yielding in the manner just described. The mean stress at failure will be f_y, i.e. failure load is not affected by the residual stress. A very slender strut will fail by buckling, i.e. $\sigma_{cr} << f_y$. For struts having intermediate slenderness, the premature yielding at the tips reduces the effective bending stiffness of the column; in this case, the column will buckle elastically at a load below the elastic critical load and the plastic squash load. The column strength curve will thus be as shown in Fig. 14.

Notice the difference between the buckling strength and the plastic squash load is most

pronounced when $\lambda = \frac{\lambda}{r} = \pi\left(E/f_y\right)^{1/2}$

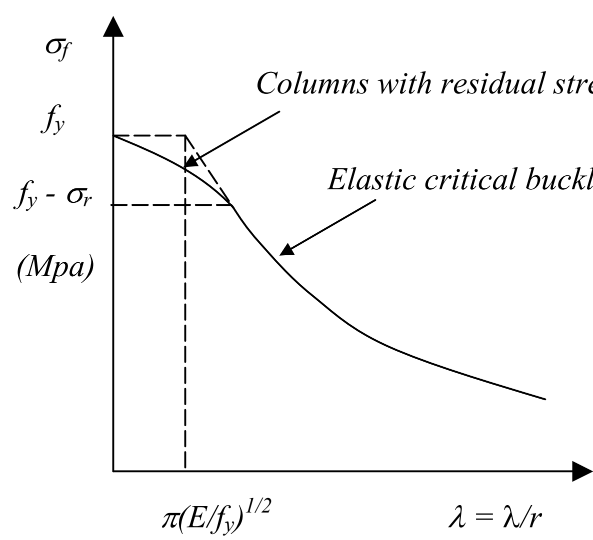


Fig. 14 Buckling of an initially straight column having residual stresses

4.4 The Effect of Strain-Hardening and the Absence of Clearly Defined Yield point

If the material of the column has a stress-strain relationship as shown in Fig. 15, the onset of first yield will not be affected, but the collapse load may be increased. Designers tend to ignore the effect of strain hardening which in fact provides a margin of safety.

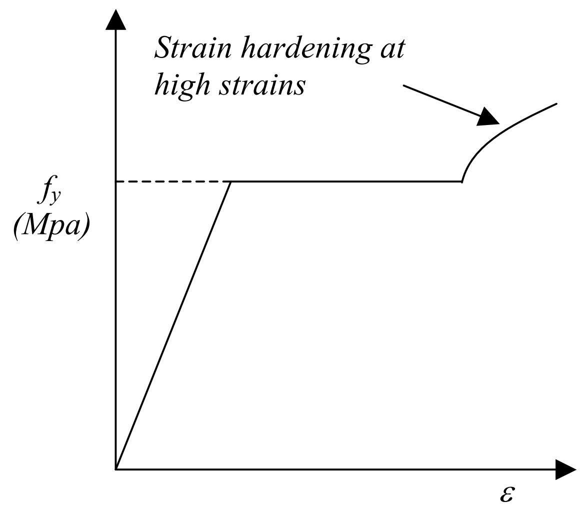


Fig. 15 Stress-strain relationship for Steels exhibiting strain hardening

High strength steels generally have stress-strain curves of the shape given in Fig. 16. At stresses above the limit of proportionality *(σ_p)*, the material behaviour is non linear and on unloading and reloading the material is linear-elastic. Most high strength structural steels Fig. 16*(a)* have an yield stress beyond which the curve becomes more or less horizontal. Some steels do not have a plastic plateau and exhibit strain-hardening throughout the inelastic range Fig. 16(*b*). In such cases, the yield stress is generally taken as the *0.2%* proof stress, for purposes of computation.

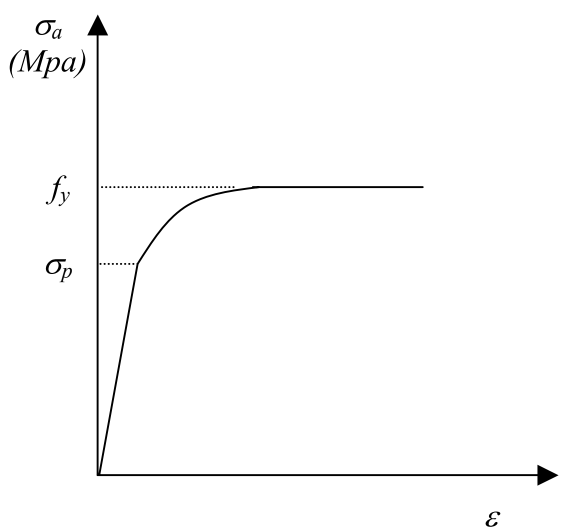


Fig.16(a)Lack of clearly defined yield

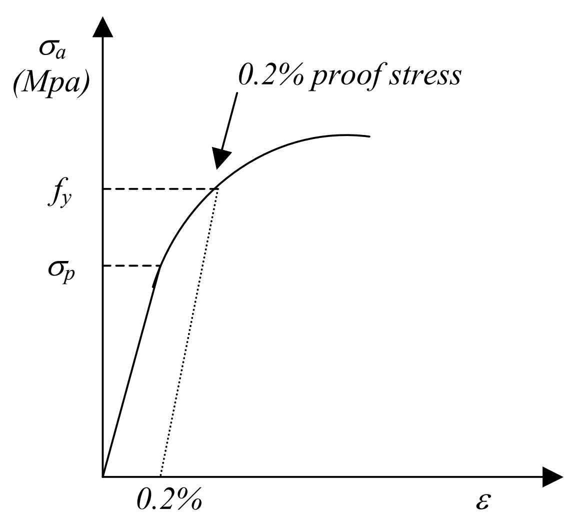


Fig.16 (b) Lack of clearly defined yield with strain hardening

4.5 The Effect of all Features Taken Together

In practice, a loaded column may experience most, if not all, of the effects listed above i.e. out of straightness, eccentricity of loading, residual stresses and lack of clearly defined yield point and strain hardening effects occurring simultaneously.

Only strain hardening tends to raise the column strengths, particularly at low slenderness values. All other effects lower the column strength values for all or part of the slenderness ratio range.

When all the effects are put together, the resulting column strength curve is generally of the form shown in Fig. 17. The beneficial effect of strain hardening at low slenderness values is generally more than adequate to provide compensation for any loss of strength due to small, accidental eccentricities in loading. Although the column strength can exceed the value obtained from the yield strength (f_y), for purposes of structural design, the column strength curve is generally considered as having a cut off at f_y, to avoid large plastic compressive deformation.

Since it is impossible to quantify the variations in geometric imperfections, accidental eccentricity, residual stresses and material properties, it is impossible to calculate with certainty, the greatest reduction in strength they might produce in practice. Thus for design purposes, it may be impossible to draw a true lower bound column strength curve. A commonly employed method is to construct a curve on the basis of specified survival probability. (For example, over *98%* of the columns to which the column curve relates, can be expected - on a statistical basis – to survive at applied loads equal to those given by the curve). All design codes provide column curves based on this philosophy. Column curves proposed for the revised Indian Code of Practice are discussed in a subsequent chapter.

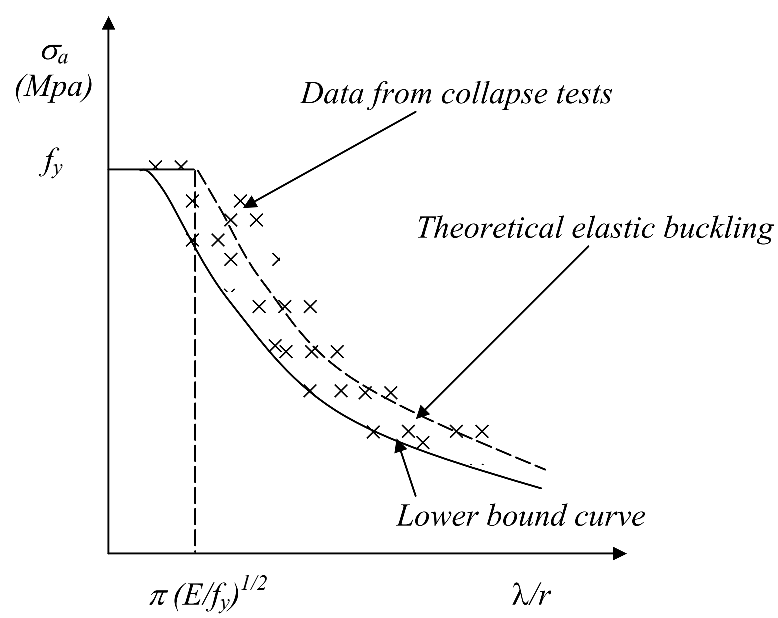


Fig. 17 General strength curves for struts with initial out of straightness,

5.0 THE CONCEPT OF EFFECTIVE LENGTHS

So far, the discussion in this chapter has been centred around pin-ended columns. The boundary conditions of a column may, however, be idealized in one the following ways

- Both the ends pin jointed (i.e. the case considered in art. 2)
- Both ends fixed.
- One end fixed and the other end pinned.
- One end fixed and the other end free.

By setting up the corresponding differential equations, expression for the critical loads as given below are obtained and the corresponding buckled shapes are given in Fig. 18.

Both ends fixed: $$P_{cr} = \frac{4\pi^2 EI}{\lambda^2} = \frac{4\pi^2 E}{\left(\frac{\lambda}{r}\right)^2}$$

One end fixed and the other end pinned: $$P_{cr} = \frac{2\pi^2 E}{\left(\frac{\lambda}{r}\right)^2}$$

One end fixed and the other end free: $$P_{cr} = \frac{\pi^2 EI}{4\lambda^2} = \frac{\pi^2 E}{4\left(\frac{\lambda}{r}\right)^2}$$

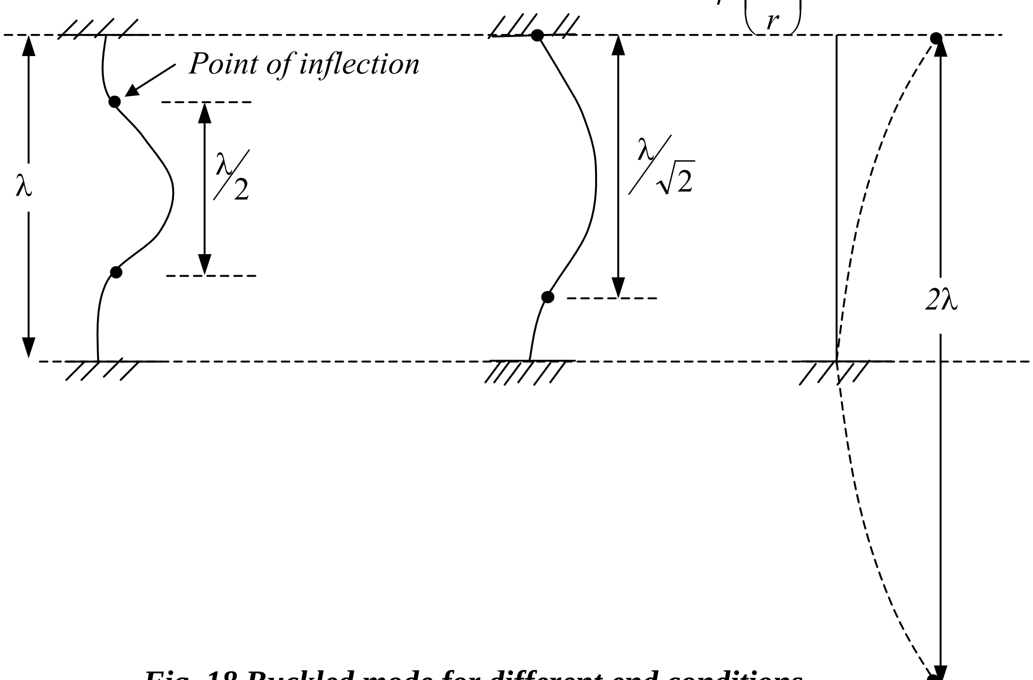


Fig. 18 Buckled mode for different end conditions

Using the column, pin ended at both ends, as the basis of comparison the critical load in all the above cases can be obtained by employing the concept of "effective length", λ_e.

It is easily verified that the calculated effective length for the various end conditions are given by

Both ends pin ended, $\lambda_e = \lambda$
Both ends fixed, $\lambda_e = \lambda / 2$

One end fixed and the other end pinned, $\lambda_e = \dfrac{\lambda}{\sqrt{2}}$

One end fixed and the other end free, $\lambda_e = 2\lambda$

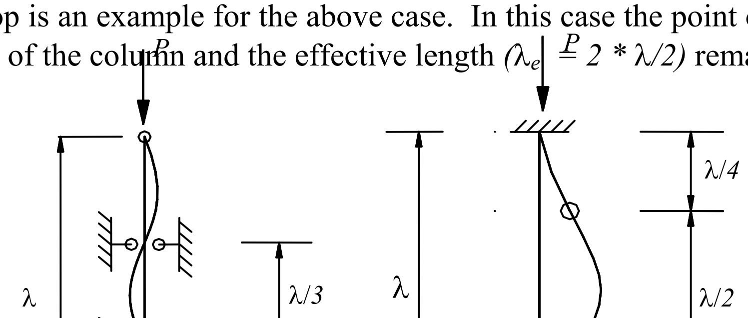
It can be seen that the effective length corresponds to the distance between the points of inflection in the buckled mode. The effective column length can be defined as the length of an equivalent pin-ended column having the same load-carrying capacity as the member under consideration. The smaller the effective length of a particular column, the smaller its danger of lateral buckling and the greater its load carrying capacity. It must be recognized that no column ends are perfectly fixed or perfectly hinged. The designer may have to interpolate between the theoretical values given above, to obtain a sensible approximation to actual restraint conditions. Effective lengths commonly employed by Designers are discussed in Chapter 10.

5.1 Effective lengths in different planes

The restraint against buckling may be different for buckling about the two column axes. Fig 19(*a*) shows a pin-ended column of *UC* section braced about the minor axis against lateral movement (but not rotationally restrained) at spacing $\lambda/3$. The minor axis buckling mode would be with an effective pin-ended column length $(\lambda_e)_y$ of $\lambda/3$. If there was no major axis bracing the effective length for buckling about the major axis $(\lambda_e)_x$ would remain as λ. Therefore, the design slenderness about the major and minor axis would be λ/r_x and $(\lambda/3)/r_y$, respectively. Generally $r_x < 3r_y$ for all UC sections, hence the major axis slenderness (λ/r_x) would be greater, giving the lower value of critical load, and failure would occur by major axis buckling. If this is not the case, checks will have to be carried out about both the axes.

Fig 19(*b*) shows a column with both ends fully restrained; the buckled shape has points of contraflexure, equivalent to pin ends, at $\lambda/4$ from either end. The central length is clearly equivalent to pin-ended column of length $\lambda/2$. This is the case, which has full rotational constraints at the ends. Fig 20 (*a*) shows the effect of partial end-restraints.

Sometimes columns are free to sway laterally, but restrained against rotation at both ends as in Fig.21 (*a*). A water tank supported on four corner columns as in Fig.21 (*b*) with rigid joints at top is an example for the above case. In this case the point of contraflexure is at mid-height of the column and the effective length $(\lambda_e = 2 * \lambda/2)$ remains λ.



5.2 Effective lengths recommended for Design

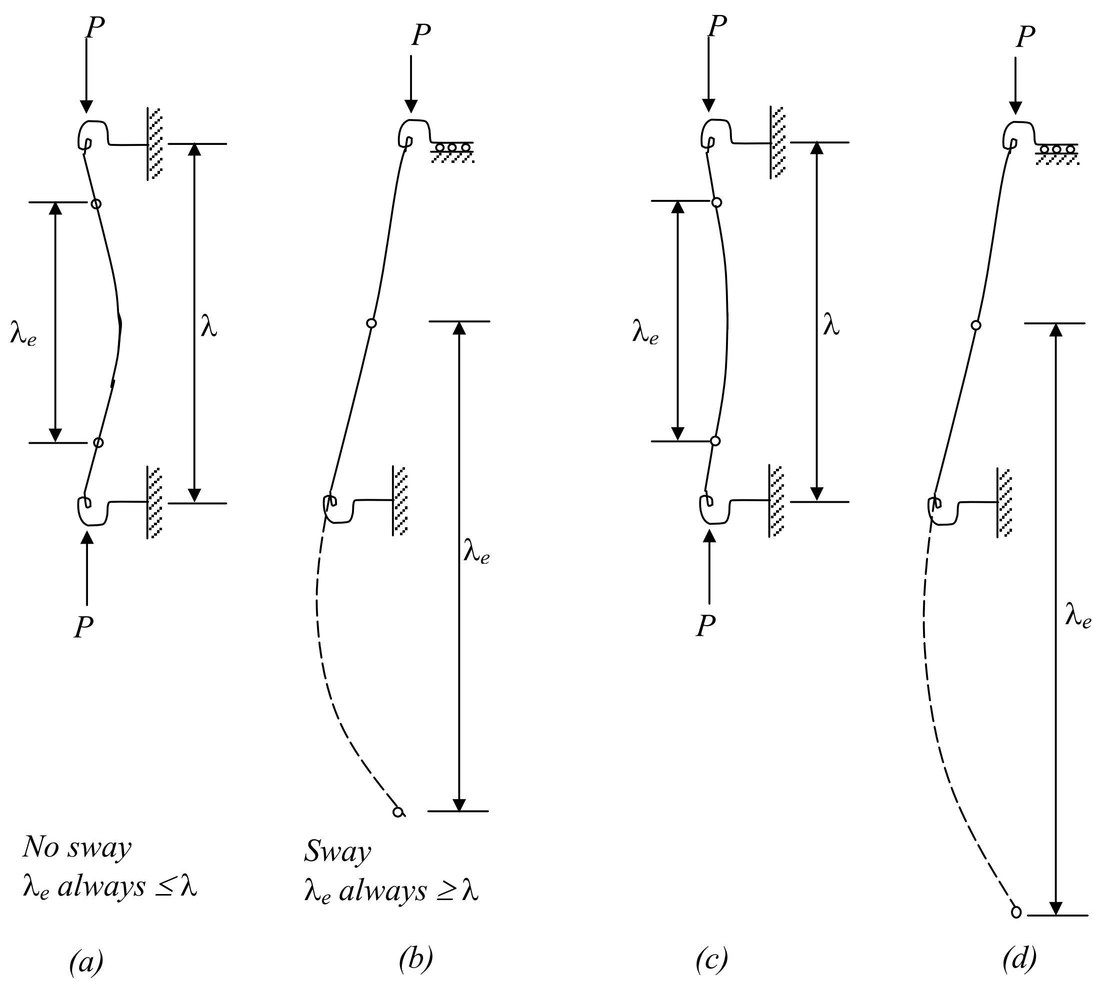


Fig. 20 Columns with partial rotational restraint

Partial end-restraints are much more common in practice than fully rigid end-constraints. The flexibility in the end-connection and (or) flexibility of the restraining members ensure partial fixity at the supports. A simple frame as shown in Fig.22 (*a*) is an example of the above case. For nodal loading, the in-plane buckling mode for this frame is shown in Fig. 22(*b*).

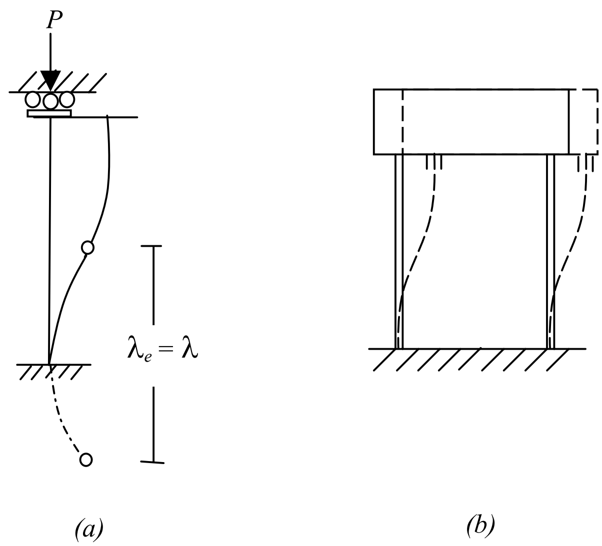


Fig.21 Columns with differing effective lengths-II

With the top beam bent in an S-shape the rotational end-restraint stiffness is given by

$$K_\theta = \frac{M}{\theta} = \frac{6EI_e}{\lambda_e}$$

For rigid beam-to-column joints this stiffness of the beam (K_θ) will control the position of the point of contraflexure in the column and thus the column effective length. These columns are represented in Fig. 22 (*c*) for which an effective length of 1.5λ is suggested.

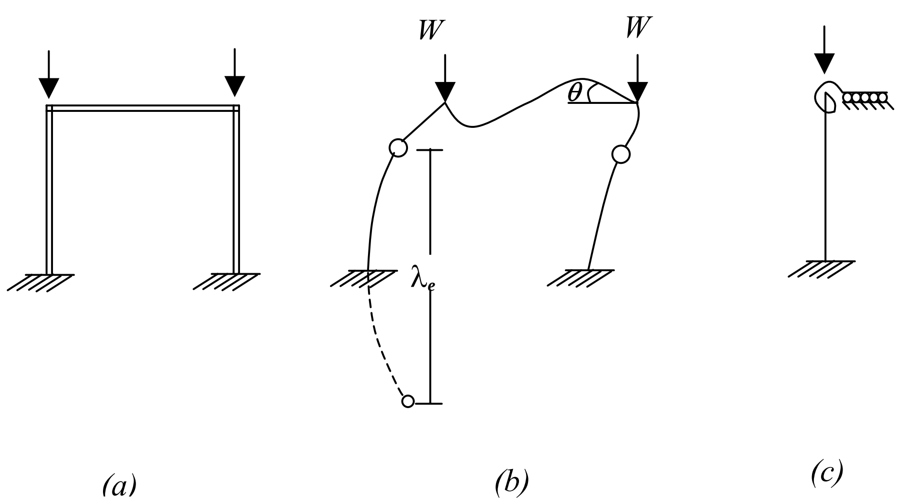


Fig. 22 Column in a simple sway frame

5.3 No-sway and sway columns

Fig. 20(*a*) and Fig. 20(*b*) represent the general cases of no-sway and sway columns with partial end-restraint. The buckled shapes will be of the form shown if the top restraint stiffness ($K_{\theta T}$) and the bottom restraint stiffness ($K_{\theta B}$) are equal. For the no sway case of Fig.20 (*a*) the position of the points of contraflexure will move within the column length as $K_{\theta T}$ and $K_{\theta B}$ vary. Fig.20*(c)* represents the situation of low $K_{\theta T}$ and high $K_{\theta B}$. However for non-sway columns λ_e is always less than or equal to λ. By contrast, for sway columns λ_e is always greater than or equal to λ. As K_θ decreases, the column end-joint rotations increase and λ_e can easily become 2λ or 3λ [Fig.20 (*d*)]. The limiting case of K_θ and $K_{\theta B} = 0$ gives $\lambda_e = \infty$.

The column design stress may be written as:

$$P_c = \frac{\pi^2 E}{\left(\lambda_e / r_y\right)^2}(area) = \left[f\left(Area\right)\right]$$

where area is dominant, the column is stocky. Otherwise the column strength is largely dependent on $(1/\lambda_e)^2$. Thus sway columns, i.e. with $\lambda_e > \lambda$, are much weaker than no-sway ones.

5.4 Accuracy in using Effective lengths

For compression members in rigid-jointed frames the effective length is directly related to the restraint provided by all the surrounding members. In a frame the interaction of all the members occurs because of the frame buckling rather than column buckling. For the design purposes, the behaviour of a limited region of the frame is considered. The limited frame comprises the column under consideration and each immediately adjacent member treated as if it were fixed at the far end. The effective length of the critical column is then obtained from a chart which is entered with two coefficients k_1, and k_2, the values of which depends upon the stiffnesses of the surrounding members k_u, k_{TL} etc. Two different cases are considered viz. columns in non-sway frames and columns in sway frames. All these cases as well as effective length charts are shown in Fig.23. For the former, the effective lengths will vary from 0.5 to 1.0 depending on the values of k_1 and k_2, while for the latter, the variation will be between 1.0 and ∞ . These end points correspond to cases of: (1) rotationally fixed ends with no sway and rotationally free ends with no sway; (2) rotationally fixed ends with free sway and rotationally free ends with free sway.

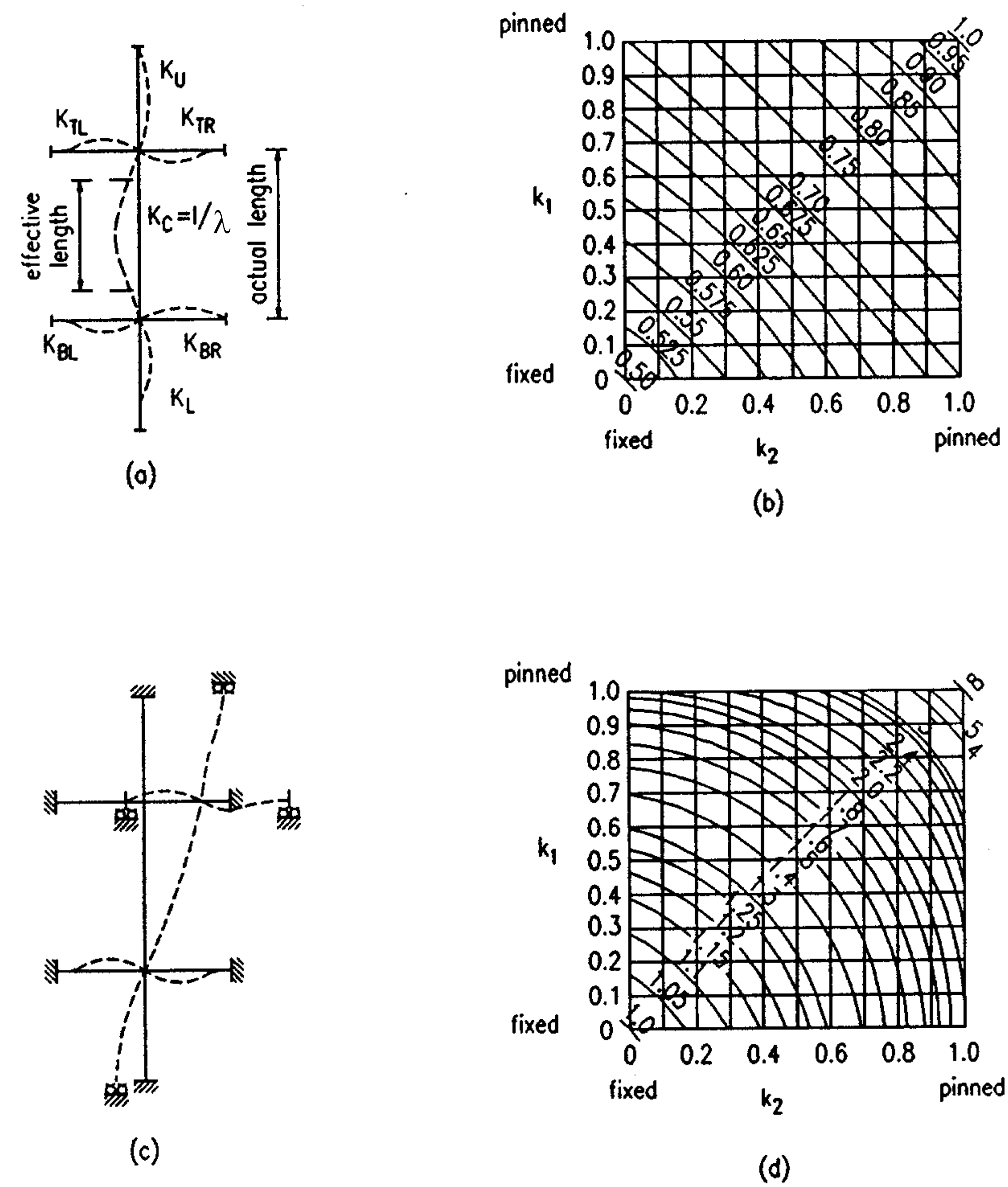


Fig. 23 Limited frames and corresponding effective length charts of BS5950: Part 1 and IS: 800.
(a) Limited frame and (b) effective length ratios ($k_3 = \infty$), for non-sway frames.
(c) Limited frame and (d) effective length ratios (without partial bracing, $k_3 = 0$), for sway frames.

6.0 TORSIONAL AND TORSIONAL-FLEXURAL BUCKLING OF COLUMNS

We have so far considered the flexural buckling of a column in which the member deforms by bending in the plane of one of the principal axes. The same form of buckling will be seen in an initially flat wide plate, loaded along its two ends, the two remaining edges being unrestrained. [See Fig. 24 (*a*)]

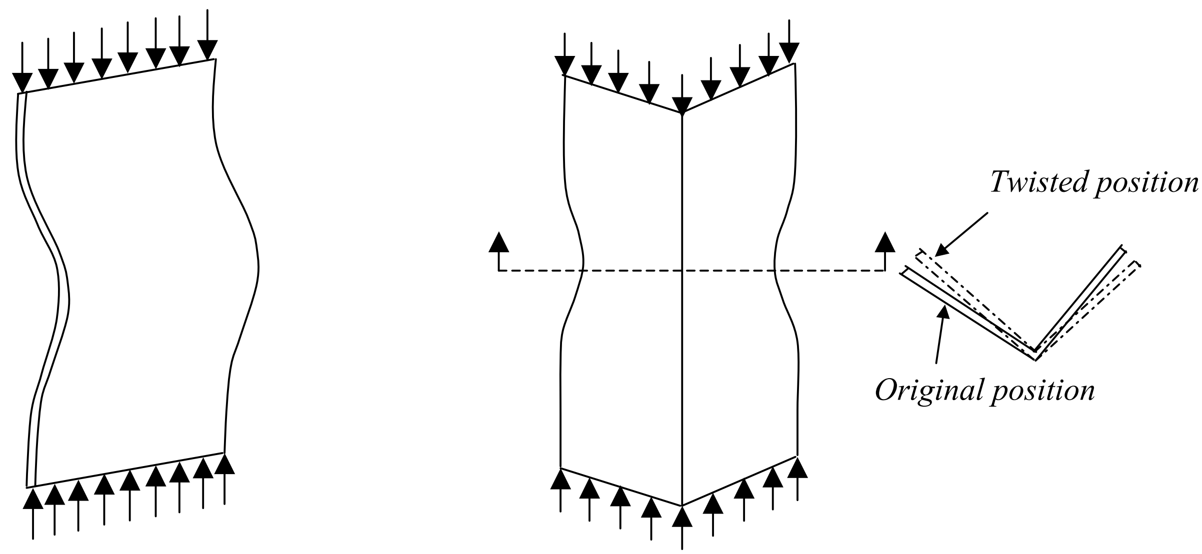


Fig.24 (a) Plate with unsupported edges ***Fig.24 (b) Folded plate twists under axial load***

On the other hand, if the plate is folded at right angles along the vertical centre-line, the resulting angular cross-section has a significantly enhanced bending stiffness. Under a uniform axial compression, the two unsupported edges tend to wave in the Euler type buckles. At the fold, the amplitude of the buckle is virtually zero. A horizontal cross-section at mid height of the strut shows that the cross-section rotates relative to the ends. This mode of buckling is essentially torsional in nature and is initiated by the lack of support at the free edges. This case illustrates buckling in torsion, due to the low resistance to twisting of the member.

Thus the column curves of the type discussed in Fig. 17 (see section 4.5) are only satisfactory for predicting the mean stress at collapse, when the strut buckles by bending in a plane of symmetry of the cross section, referred to as " flexural buckling ". Members with low torsional stiffness (eg. angles, tees etc made of thin walled members) will undergo torsional buckling before flexural buckling. Cruciform sections are generally prone to torsional buckling before flexural buckling. Singly symmetric or un-symmetric cross sections may undergo combined twisting about the shear centre and a translation of the shear centre. This is known as " torsional – flexural buckling ".

In this article we shall determine the critical load of columns that buckle by twisting or by a combination of both bending and twisting. The investigation is limited to open thin-walled sections as they are the only sections that are susceptible to torsional or torsional-flexural buckling. The study is also restricted to elastic behaviour, small deformations and concentric loading. The critical load is determined either by integrating the governing differential equations or by making use of an energy principle. The analysis presented here uses the Rayleigh-Ritz energy method to determine the critical load.

Let us consider the thin-walled open cross-section of arbitrary shape given in Fig. 25. The deformation taking place during buckling is assumed to consist of a combination of twisting and bending about two axis. To express strain energy in its simplest form the deformation is reduced to two pure translations and a pure rotation. The origin '*O*' is assumed to be the shear centre. The *x* and *y* directions are assumed to coincide with the principal axis of the section, and the *z* direction is taken along longitudinal axis through shear centre, *O*.

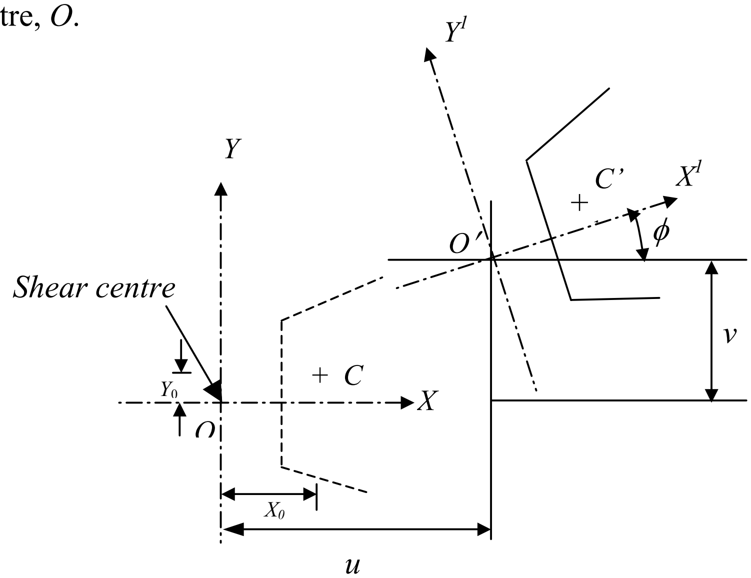


Fig. 25 Torsional -flexural buckling deformations.

(Note: In deriving Euler equations, we used x axis along the length of column; here we are using z axis along column length)

The co-ordinates of the centroid are denoted by x_o and y_o . As a result of buckling the cross section undergoes translations u and v in the x and y directions respectively, and rotation ϕ about the z-axis. The geometric shape of the cross section in the xy plane is assumed to remain undisturbed throughout.

Boundary conditions:

It is assumed that the displacements in the x and y directions and the moments about these axis vanish at the ends of the member. That is,
$u = v = 0$ at $z = 0$ and λ

$$\frac{d^2u}{dz^2} = \frac{d^2v}{dz^2} = 0 \;\; at \;\; z=0 \;\; and \; \lambda \tag{9}$$

The torsional conditions which correspond to these flexural conditions are zero rotation and zero warping restraint at the ends of the member. Thus

$$\phi = \frac{d^2\phi}{dz^2} = 0 \quad at \quad z = 0 \ and \ \lambda \tag{10}$$

The boundary conditions will be satisfied by assuming a deflected shape of the form

$$u = C_1 \ Sin\frac{\pi z}{\lambda}$$

$$v = C_2 \ Sin\frac{\pi z}{\lambda}$$

$$\phi = C_3 \ Sin\frac{\pi z}{\lambda} \tag{11}$$

Strain energy stored in the member consists of four parts. Those are

i. energy due to bending in x-direction
ii. energy due to bending in y-direction
iii. energy of the St.Venant shear stresses.
iv. energy of the longitudinal stresses associated with warping torsion.

Thus total strain energy is given by

$$U = \frac{1}{2}\int_0^\lambda EI_y\left(\frac{d^2u}{dz^2}\right)^2 dz + \frac{1}{2}\int_0^\lambda EI_x\left(\frac{d^2v}{dz^2}\right)^2 dz$$

$$+ \frac{1}{2}\int_0^\lambda GJ\left(\frac{d\phi}{dz}\right)^2 dz + \frac{1}{2}\int_0^\lambda E\Gamma\left(\frac{d^2\phi}{dz^2}\right)^2 dz \tag{12}$$

where J and Γ are the torsional constant and warping constant of the section respectively.

Substitution of the assumed deflection function (Eqn. 11) into the strain energy expression (Eqn. 12) and then simplification gives

$$U = \frac{1}{4}\frac{\pi^2}{\lambda}\left[C_1^2\ \frac{EI_y\pi^2}{\lambda^2} + C_2^2\ \frac{EI_x\pi^2}{\lambda^2} + C_3^2\left(GJ + \frac{E\Gamma\ \pi^2}{\lambda^2}\right)\right] \tag{13}$$

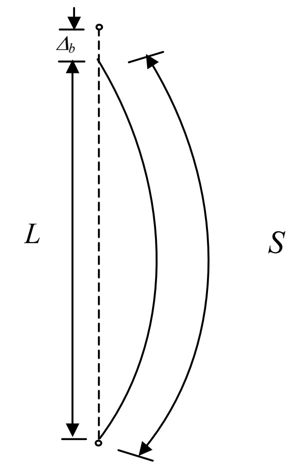
Potential Energy:

The potential energy of the external loads is equal to the negative product of the loads and the distances they move as the column deforms. Potential energy is given by

$$V = -\int_A \Delta_b \, \sigma \, dA \tag{14}$$

where dA is the cross sectional area of the fibre and the load it supports is $\sigma \, dA$
Δ_b is equal to the difference between the arc lengths and the chord length L of the fibre.
i.e. $\Delta_b = S - L$ (Fig. 26) (15)

Fig. 26 Axial shortening of longitudinal fibre due to bending



The potential energy of the external loads can be shown to be given by

$$V = -\frac{P\pi^2}{4\lambda}\left(C_1^2 + C_2^2 + C_3^2 r_0^2 - 2C_1 C_3 y_0 + 2C_2 C_3 x_0\right) \tag{16}$$

where, x_o and y_o are the co-ordinates of centroid and r_o is the polar radius of gyration.

Total potential energy of the system is

$$U + V = \frac{\pi^2}{4\lambda}\left\{\begin{array}{l} C_1^2\left(\dfrac{\pi^2 EI_y}{\lambda^2} - P\right) + C_2^2\left(\dfrac{\pi^2 EI_x}{\lambda^2} - P\right) + C_3^2 r_0^2\left(\dfrac{1}{r_0^2}\left(GJ + \dfrac{E\Gamma\pi^2}{\lambda^2}\right) - P\right) \\ \quad + 2C_1 C_3 P y_0 - 2C_2 C_3 P x_0 \end{array}\right\} \tag{17}$$

substituting, $$P_y = \frac{\pi^2 EI_y}{\lambda^2};\ P_x = \frac{\pi^2 EI_x}{\lambda^2} \quad and\ P_\phi = \frac{1}{r_0^2}\left(GJ + \frac{E\Gamma\pi^2}{\lambda^2}\right)$$

Thus, equation (17) becomes

$$U + V = \frac{\pi^2}{4\lambda}\left[C_1^2\left(P_y - P\right) + C_2^2\left(P_x - P\right) + C_3^2 r_0^2\left(P_\phi - P\right) + 2C_1 C_3 P y_0 - 2C_2 C_3 P x_0\right] \tag{18}$$

Since, *(U+V)* is a function of three variables, it will have a stationary value when its derivatives with respect to C_1, C_2 and C_3 vanish. Thus,

$$\frac{\partial(U+V)}{\partial C_1} = C_1\left(P_y - P\right) + C_3\left(Py_0\right) \qquad = 0$$

$$\frac{\partial(U+V)}{\partial C_2} = C_2\left(P_x - P\right) - C_3\left(Px_0\right) \qquad = 0$$

$$\frac{\partial(U+V)}{\partial C_3} = C_1\, Py_0 - C_2\, Px_0 + C_3\, r_0^2\left(P_\phi - P\right) \qquad = 0 \tag{19}$$

$$\begin{bmatrix} P_y - P & 0 & Py_0 \\ 0 & P_x - P & -Px_0 \\ Py_0 & -Px_0 & r_0^2(P_\phi - P) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{20.a}$$

The solution to this equation could be found by setting the determinant to be zero.

$$\begin{vmatrix} P_y - P & 0 & Py_0 \\ 0 & P_x - P & -Px_0 \\ Py_0 & -Px_0 & r_0^2(P_\phi - P) \end{vmatrix} = 0 \tag{20.b}$$

Hence, the critical load is determined by the equation,

$$\left(P_y - P\right)\left(P_x - P\right)\left(P_\phi - P\right) - \left(P_y - P\right)\frac{P^2 x_0^2}{r_0^2} - \left(P_x - P\right)\frac{P^2 y_0^2}{r_0^2} = 0 \tag{21}$$

This is a cubic equation in P; the three roots of the cubic equation are the critical loads of the member, corresponding to the three buckling mode shapes.

a) For cross-section with double symmetry the centroid and shear centre coincide, hence

$x_0 = 0$ and $y_0 = 0$

$$\therefore\ (P_y - P)(P_x - P)(P_\phi - P) = 0 \tag{22}$$

This equation has three roots, namely,

$$\left.\begin{aligned} P &= P_x = \frac{\pi^2 EI_x}{\lambda^2} \\ P &= P_y = \frac{\pi^2 EI_y}{\lambda^2} \end{aligned}\right\} \quad - \text{ These represent Euler loads by buckling about the } x$$

and y axes

$$P = P_\phi = \frac{1}{r_0^2}\left(GJ + \frac{E\Gamma\pi^2}{\lambda^2} \right) \quad - \text{ This represents the Torsional buckling load.} \qquad (23)$$

Depending on the cross sectional property of the member any of the critical load values would govern.

b) For singly symmetric sections (such as channel sections):-

When the cross-section has only one axis of symmetry, say the x-axis,(eg. a channel section) the shear centre will be on that axis, hence equation (22) becomes a quadratic equation,

$$y_0 = 0 \qquad P = P_y = \frac{\pi^2 EI_y}{\lambda^2} \quad \text{(This represents Euler Buckling Load)} \qquad (24)$$

$$\therefore \left(P_x - P\right)\left(P_\phi - P\right) - \frac{P^2 x_0^2}{r_0^2} = 0 \qquad (25)$$

This quadratic equation in P has two roots, which correspond to flexural-torsional buckling.

The smaller root of the above equation is

$$P_{TF} = \frac{1}{2k}\left[P_\phi + P_x - \sqrt{\left(P_\phi + P_x\right)^2 - 4kP_\phi P_x} \right] \qquad (26)$$

in which $k = \left[1 - \left({x_0}/{r_0} \right)^2 \right]$

and P_{TF} is torsional-flexural buckling load.

Thus a singly symmetric section such as an equal angle or a channel can buckle either by flexure in the plane of symmetry or by a combination of flexure and torsion. ***All centrally loaded columns have three distinct buckling loads, at least one of which corresponds to torsional or torsional - flexural mode in a doubly symmetric section. Flexural buckling load about the weak axis is almost always the lowest. Hence, we disregard the torsional buckling load in doubly symmetric sections. In non-symmetric sections, buckling will be always in torsional – flexural mode regardless of its shape and***

dimensions. However, non-symmetric sections are rarely used and their design does not pose a serious problem.

Thin-walled open sections, such as angles and channels, can buckle by bending or by a combination of bending and twisting. Which of these two modes is critical depends on the shape and dimensions of the cross-section. Hence, torsional-flexural buckling must be considered in their design.

7.0 CONCLUDING REMARKS

The elastic buckling of an ideally straight column pin ended at both ends and subjected to axial compression was considered. The elastic buckling load was shown to be dependent on the slenderness ratio (λ/r) of the column. Factors affecting the column strengths (viz. initial imperfection, eccentricity of loading, residual stresses and lack of well-defined elastic limit) were all individually considered. Finally a generalized column strength curve (taking account of all these factors) has been suggested, as the basis of column design curves employed in Design Practices. The concept of "effective length" of the column has been described, which could be used as the basis of design of columns with differing boundary conditions.

The phenomenon of Elastic Torsional and Torsional-flexural buckling of a perfect column were discussed conceptually. The instability effects due to torsional buckling of slender sections are explained and discussed. Applications to doubly and singly symmetric sections are derived.

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